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CENTRE DE RECHERCHE SUR LES TRANSPORTS

CASE POSTALE 6128, SUCCURSALE A, MONTRÉAL, QUÉBEC, H3C 3J7

OPTIMAL WEIGHTED INFLUENCE SPHERES IN NETWORKS*

by

Horst A. Eiselt[1]
and
Gilbert Laporte[2]

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[1] Faculty of Administration, University of New Brunswick, Fredericton, New Brunswick, Canada E3B 5A3.

[2] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, Succursale A, Montréal, Canada H3C 3J7.

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ABSTRACT

In this paper, a conditional weighted location problem on a network is examined. Based on a specific attraction function, influence spheres of facilities are defined. We describe a method which simultaneously locates an additional facility on the given network and optimizes its weight. A second model is then delineated which avoids some of the shortcomings of the first model.

Key words : Voronoi diagrams, influence spheres, networks, location, weights.

RÉSUMÉ

Dans cet article, on décrit un problème conditionnel de localisation d'installations dans un réseau, en présence de poids. On définit les sphères d'influence des installations à partir d'une fonction d'attraction. On décrit en premier lieu un modèle qui permet de déterminer simultanément la localisation et le poids optimaux d'une nouvelle installation. Par la suite, on procède à l'élaboration d'un second modèle qui corrige certaines carences du premier.

Mots-clés : diagrammes de Voronoi, sphères d'influence, réseaux, localisation, poids.

1. Introduction

The purpose of this paper is to develop a model whose solution simultaneously optimizes the location of a new facility on a given network as well as one of its features called its weight. This work is rooted in closest point models usually known as Voronoi diagrams and in gravity models. The original work on Voronoi diagrams is due to Voronoi (1908) and Thiessen (1911); for a recent survey see Eiselt and Pederzoli (1986). Voronoi diagrams have a large number of applications in fields as diverse as archaeology, physics, geography and biology (for pertinent references see Aurenhammer and Edelsbrunner (1984)) and VLSI design (Meshkat and Sakkas (1987)). A variety of related problems as well as optimal algorithms for the construction of a Voronoi diagram in the plane are given by Shamos and Hoey (1976) and recently Fortune (1987).

The other foundation of this study derives from the gravity models first introduced and discussed by Reilly (1929) and Converse (1948) and later further developed by Huff (1964) and (1973). Huff's "trading areas" coincide with the Voronoi areas described earlier. To our knowledge it was Boots (1980) who introduced the weights long known to "retail gravitationalists" into the context of Voronoi diagrams. The algorithm for the determination of weighted Voronoi diagrams was developed by Aurenhammer and Edelsbrunner (1984). Models optimizing facility location and weights in the context of one-dimensional Voronoi diagrams have been studied by Eiselt and Laporte (1987) and (1988). The model investigated by ReVelle (1986) is simpler in that it uses equal weights for the facilities but it finds locations on a network which is a more complex space than the Eiselt and Laporte linear market.

In this study, we will generalize ReVelle's findings in two directions. Firstly, we allow differential weights for the facilities and

secondly, we will leave a parameter of the cost function variable so as to facilitate sensitivity analysis of the optimal solution. The results of the basic model exhibit the rather undesirable property of "jumps" in the profit function. Not only are these jumps undesirable but they also appear unrealistic. On this basis, an alternate model is suggested and demonstrated in a numerical example.

2. The Basic Model

Consider a graph $G = (N,A,b,d)$ with sets of nodes $N = \{n_1, \ldots, n_n\}$ and arcs $A = \{a_{ij}: n_i, n_j \in N, i \neq j\}$. The mapping $b:N \to \mathbb{R}_+^n$ is the demand function where b_i specifies the demand (in dollars) for a homogeneous product at node n_i and the mapping $d:A \to \mathbb{R}_+^s$ is the distance function with d_{ij} being the length of a shortest path from n_i to n_j. Let now the scalar $w_j \geq 0$ denote the weight of a facility located at n_j. This weight measures parameters as diverse as the size of the facility, its relative price advantage, the courteousness of its staff, etc. In general, a facility with a large weight is more popular among consumers than one with a small weight. If a node n_j is not occupied by a facility, then $w_j = 0$. In our model the given graph G may be directed, undirected, or mixed. Note however that directed arcs symbolizing one-way streets may be meaningless if different means of transportation are allowed: a one-way street is unidirectional for automobile traffic but it may be passed through in either direction by neighborhood shoppers. For simplicity, we assume a unique transportation mode.

It is usually assumed that customers patronize the facility which is closest to them. This assumption is justified only as long as the different facilities are comparable in their prices, convenience and other features. If differences between facilities become non-negligable, a parameter measuring these differences, here called weight, has to be introduced.

Moreover, a function has to be designed measuring the trade-off between the weight of a facility and the facility-customer distance, as perceived by the customer. Here it is assumed that a customer at n_i is attracted to a facility located at n_j according to the attraction function $\phi_{ij} = w_j/d_{ij}$. Similar functions have been used in the aforementioned papers by Boots (1980) and Aurenhammer and Edelsbrunner (1984). The function is essentially a simplified version of the gravity-type models which work with squared distances that appear in a large variety of physical contexts. A common feature of all attraction functions is that an increasing weight increases the attraction of a customer to a facility whereas an increasing distance between them decreases it.

We are now able to define a service function $\psi_i = \max_k \{\phi_{ik}\}$. Assuming that the customers patronize exclusively the facility they are most attracted to, the influence sphere S_j of the j^{th} facility is the set of all nodes where the attraction to the j^{th} facility equals the service level, i.e. $S_j = \{n_i : \phi_{ij} = \psi_i\}$. Then the j^{th} facility faces a total demand for its services of $\sum_{i \in S_j} b_i$. If the service level is determined by more than one facility, the degenerate case, then the demand at that node is evenly split among those facilities currently offering the highest service level.

The location model studied here is – just like that investigated by ReVelle – of the conditional type: given that a set X of p facilities in the network at fixed locations, where should r new facilities be located? The solution to this problem is referred to as an (r/X_p) medianoid by Hakimi (1983). Furthermore: what should the respective weights of these new facilities be? This problem can be formulated as a mathematical program as follows. As above, N denotes the current set of nodes; let $M \subseteq N$ be the set of potential facility sites. The cost of establishing a facility of weight w_j at site $n_j \in M$ is equal to $C_j = cw_j$ where $c > 0$. Let $p_i > 0$ be the number of

facilities among which the revenue of node $n_i \in N$ is currently split and $\bar{\psi}_i$, its current service level. Also define:

r : the number of new facilities to be located

$$x_j : \begin{cases} 1 & \text{if a new facility is sited at } n_j \in M \\ 0 & \text{otherwise} \end{cases}$$

$$y_{0i} : \begin{cases} 1 & \text{if in the optimal solution, } n_i \text{ is entirely captured by at least one new facility} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ki} \ (k=1, \ldots, M) = \begin{cases} 1 & \text{if in the optimal solution, } n_i \text{ is equally attracted to } k \text{ new facilities on top of the facilities to which it is currently most attracted} \\ 0 & \text{otherwise} \end{cases}$$

Then, the problem is:

$$\text{maximize} \quad \sum_{i \in N} \left[b_i \, y_{0i} + \sum_{k=1}^{|M|} \left(\frac{k b_i}{k+p_i} \right) y_{ki} \right] - c \sum_{j \in M} w_j$$

subject to

$$y_{0i} \leq \sum_{j \in M_i} x_j \qquad (i \in N) \qquad (1)$$

$$k y_{ki} \leq \sum_{j \in Q_i} x_j \qquad (i \in N;\ k=1, \ldots, |M|) \qquad (2)$$

$$\sum_{k=0}^{|M|} y_{ki} \leq 1 \qquad (i \in N) \qquad (3)$$

$$\sum_{j \in M} x_j = r \qquad (4)$$

$$M_i = \{ j \in M : w_j / d_{ij} > \bar{\psi}_i \} \qquad (i \in N) \qquad (5)$$

$$Q_i = \{j \in M : w_j/d_{ij} = \bar{\psi}_i\} \qquad (i \in N) \qquad (6)$$

$$0 \leq w_j \leq Tx_j, \; T > 0 \qquad (j \in M) \qquad (7)$$

$$x_j = 0 \text{ or } 1 \qquad (j \in M) \qquad (8)$$

$$y_{ki} = 0 \text{ or } 1 \qquad (i \in N;\; k=0, \ldots, |M|) \qquad (9)$$

In this model, the first summation of the objective function represents the total revenue captured by the new facilities, whereas the last term corresponds to the cost of establishing these facilities. Constraints (1) and (5) guarantee that y_{0i} can be equal to 1 only if a facility established at some site $j \in M$ is such that $w_j/d_{ij} > \bar{\psi}_i$. Similarly, constraints (2) and (6) state that if the revenue of node n_i is to be equally split between k new facilities, on top of the existing p_i facilities, then the number of sites that will receive these facilities must be at least equal to k. Constraints (3) ensure that at most one of the above situations can occur. Among all variables y_{ki} (k=0, ..., $|M|$) which can take the value 1, for a given i, the variable having the largest coefficient in the objective function will be selected. Since

$$b_i > \frac{kb_i}{k+p_i} > \frac{k'b_i}{k'+p_i} \qquad \text{for } k > k' \geq 1 \text{ and } p_i > 0,$$

the proper revenue count is ensured. In constraints (7), T is an arbitrarily large number; these constraints stipulate that a weight can only be assigned to nodes where facilities are located.

For the type of attraction function used here, this model cannot be solved easily by any of the known integer programming codes. This is so because constraints (1) and (2) are conditional upon the definition of the sets M_i and Q_i (constraints (5) and (6)) which in turn depend on the decision variables w_j. Linearizing these conditions, along the lines of Dantzig (1960) for example, would be cumbersome and impractical.

A simple case of the model occurs when the w_j's are known constants or are not used to define the attraction function. Then, these variables can be dropped from the objective function, the sets M_i and Q_i can easily be defined a priori, constraints (7) can be removed and the variables y_{ki} need only be defined for $k=1, \ldots, |Q_i|$. ReVelle (1986) used such a formulation.

In this paper, we explicitly include facility weights in the attraction function, but we consider the case where only one additional facility is to be introduced, i.e. $r=1$. We will show that this case can be solved quite easily in polynomial time.

3. Solution of the Basic Model

We will approach the problem by tentatively locating the new facility at some node, say n_j, find its optimal weight and corresponding profit and then repeat these computations for all admissible locations. Consider now the node n_i, its service level $\bar{\psi}_i$ and let $n_i \in S_v$ for some v. Then customers at n_i will be equally attracted to the new facility at n_j whenever $\phi_{ij} = \bar{\psi}_i$ or $w_j = \bar{\psi}_i d_{ij}$ which is the threshold value. In particular, if $w_j < \bar{\psi}_i d_{ij}$, then the entire demand at n_i is served by the facility at n_v as before; if $w_j = \bar{\psi}_i d_{ij}$, half of the demand at n_i is served by the v^{th} and the other half by the j^{th} facility (the degenerate case) and if $w_j > \bar{\psi}_i d_{ij}$, the new facility serves the entire demand at n_i. In case of i=j=v, the threshold value is simply w_v. For a tentatively fixed location of the new facility at n_j, each demand node n_i generates such a threshold value which leads to a breakpoint in the revenue function as shown below. Clearly, if a facility is already located at n_i the service level at that node approaches infinity and so does the breakpoint. The result is a maximal number of n breakpoints. As above, the revenue of a new facility at n_j is $R_j = \sum_{i \in S_j} b_i$, the cost is $C_j = cw_j$ and the profit function is $P_j = R_j - C_j$.

The following numerical example may illustrate this. Consider the graph G = (N,A,b,d') shown in Figure 1:

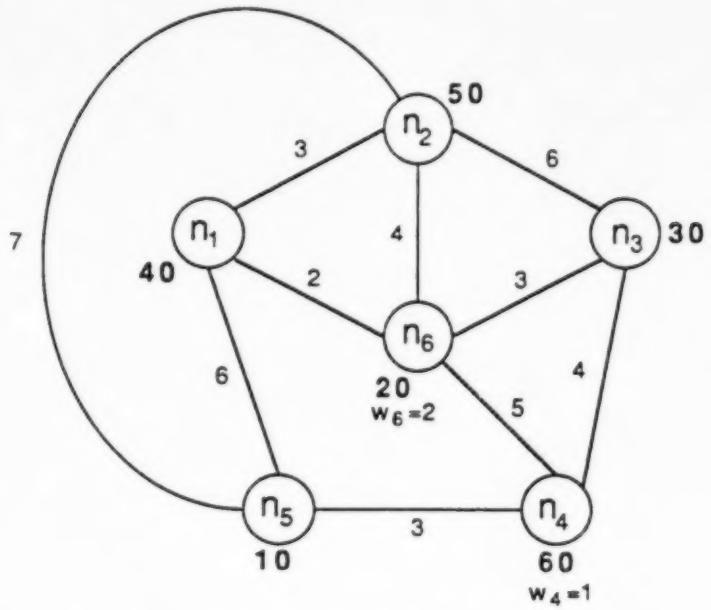

Figure 1 : A six node graph with two facilities

The numbers next to the nodes are the demands and the numbers on the edges represent the lengths of the direct connections d'_{ij} ($=d'_{ji}$) from which the lengths d_{ij} of the shortest paths are calculated. Currently, one facility with weight one exists at n_4 and a facility with weight two exists at n_6. The attraction and service functions are summarized in Table 1.

Table 1 : Attraction and service function for the six node example

n_i	n_1	n_2	n_3	n_4	n_5	n_6
ϕ_{i4}	1/7	1/9	1/4	∞	1/3	1/5
ϕ_{i6}	1	1/2	2/3	2/5	1/4	∞
ψ_i	1	1/2	2/3	∞	1/3	∞
node currently serving n_i	n_6	n_6	n_6	n_4	n_4	n_6

Suppose now that we tentatively locate a new facility at n_2. For the time being, let its weight w_2 be unspecified. Calculating the breakpoints for the nodes n_i, i=1, ..., 6, as $w_i d_{i2}$, we obtain 3, 0, 4, ∞, 7/3, ∞, respectively. Thus the revenue function can be written as

$$R_2 = \begin{cases} 0 & \text{if } w_2 = 0 \\ 50 & \text{if } w_2 \in]0;7/3[\\ 55 & \text{if } w_2 = 7/3 \\ 60 & \text{if } w_2 \in]7/3;3[\\ 80 & \text{if } w_2 = 3 \\ 100 & \text{if } w_2 \in]3;4[\\ 115 & \text{if } w_2 = 4 \\ 130 & \text{if } w_2 \in]4;\infty[\end{cases}$$

This revenue function is displayed in Figure 2a. Assuming a linear cost function $C_2 = 27w_2$ we obtain the profit function in Figure 2b.

The sawtooth function depicted in Figure 2b is the typical profit function in our model. It indicates that all local maxima are located just to the right of the breakpoints, in our case at $\bar{w}_2 = \theta$, $7/3+\theta$, $3+\theta$, and $4+\theta$ with $\theta > 0$ but arbitrarily small. This result is rather general: it remains valid as long as the cost function remains monotonously non-decreasing (which is quite a weak assumption).

For the sake of a subsequent sensitivity analysis let the coefficient c be unspecified. The profit function is then $P_2 = \max\{50-\theta c, 10-7\theta/3-\theta c, 100-3c-\theta c, 130-4c-\theta c\}$. Examining this function for all nonnegative values of c, we obtain the following decision rule: if $c \in [0;20]$, the optimal weight is $\bar{w}_2 = 4+\theta$ and the actual profit is $\bar{P}_2 = 130-4c-\theta c$; if $c \in [20;\infty]$, then $\bar{w}_2 = \theta$ and $\bar{P}_2 = 50-\theta c$.

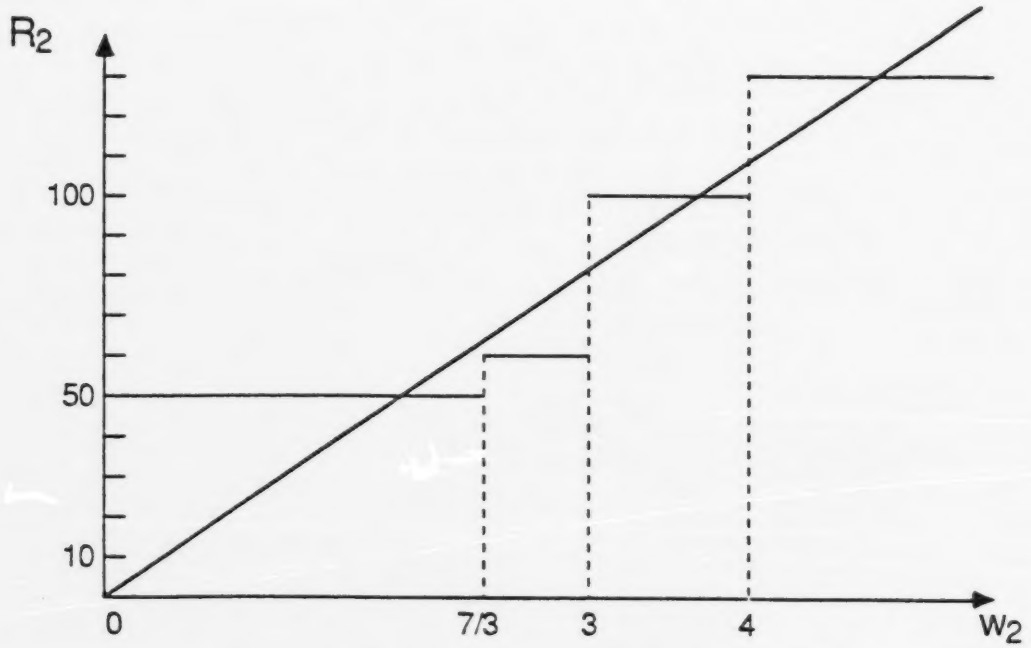

Figure 2a : Revenue as a function of weight, for a facility located at node n_2.

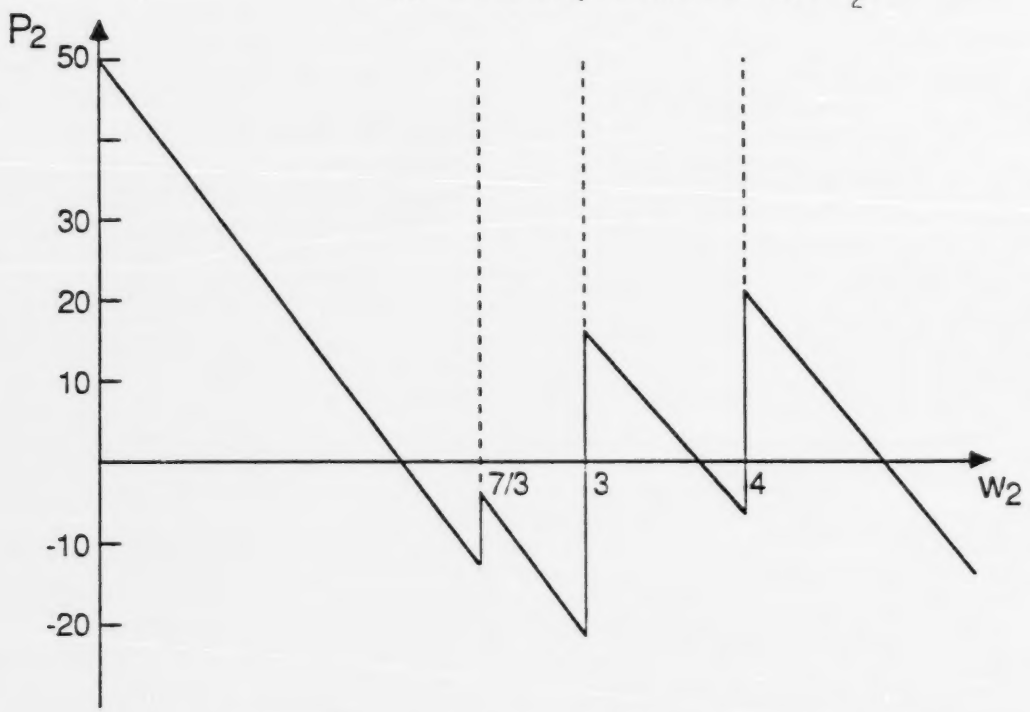

Figure 2b : Profit as a function of weight, for a facility located at node n_2.

This analytical result makes intuitive sense: if the costs are small, the new facility will carry a large weight whereas high variable costs result in a small neighborhood-type facility.

If the tentative location for the new facility is not the only feasible location, then the above procedure will have to be repeated for all admissible locations. For each of these locations the profit function $\bar{P}(c)$ will be piecewise linear and convex. If this is done in our example we obtain the functions displayed in Figure 3.

The upper envelope of these profit functions specifies the optimal locations as well as the corresponding profits. In this example we obtain:

if $c\varepsilon[0;20]$: locate at n_4 with $\bar{w}_4 = 9/2+\theta$ resulting in $\bar{P}_4 = 190-9c/2-\theta c$,
if $c\varepsilon[20;45]$: locate at n_6 with $\bar{w}_6 = 2+\theta$ resulting in $\bar{P}_6 = 140-2c-\theta c$, and
if $c\varepsilon[45;\infty[$: locate at n_2 with $\bar{w}_2 = \theta$ resulting in $\bar{P}_2 = 50-\theta c$.

Note that locating at n_1 or n_2 dominates locating at n_3 for all $c>0$; similarly locating at n_3 dominates locating at n_5 for all $c>0$. These dominances create a useful hierarchy in the decision-making process. The boundaries in the above decision rule are reasonably wide, thus for costs which are well within one of the intervals and which do not fluctuate much, the decision is likely to remain optimal.

The computational complexity of the procedure is easily determined. For each tentative location of the new facility, the profit function consists of $O(n)$ breakpoints each of which can be evaluated in constant time. Since there are $O(n)$ tentative locations, the procedure is $O(n^2)$. Note that this argument assumes that the shortest distance matrix is readily available. If not, it has to be determined in $O(n^3)$ time which then dominates the complexity function.

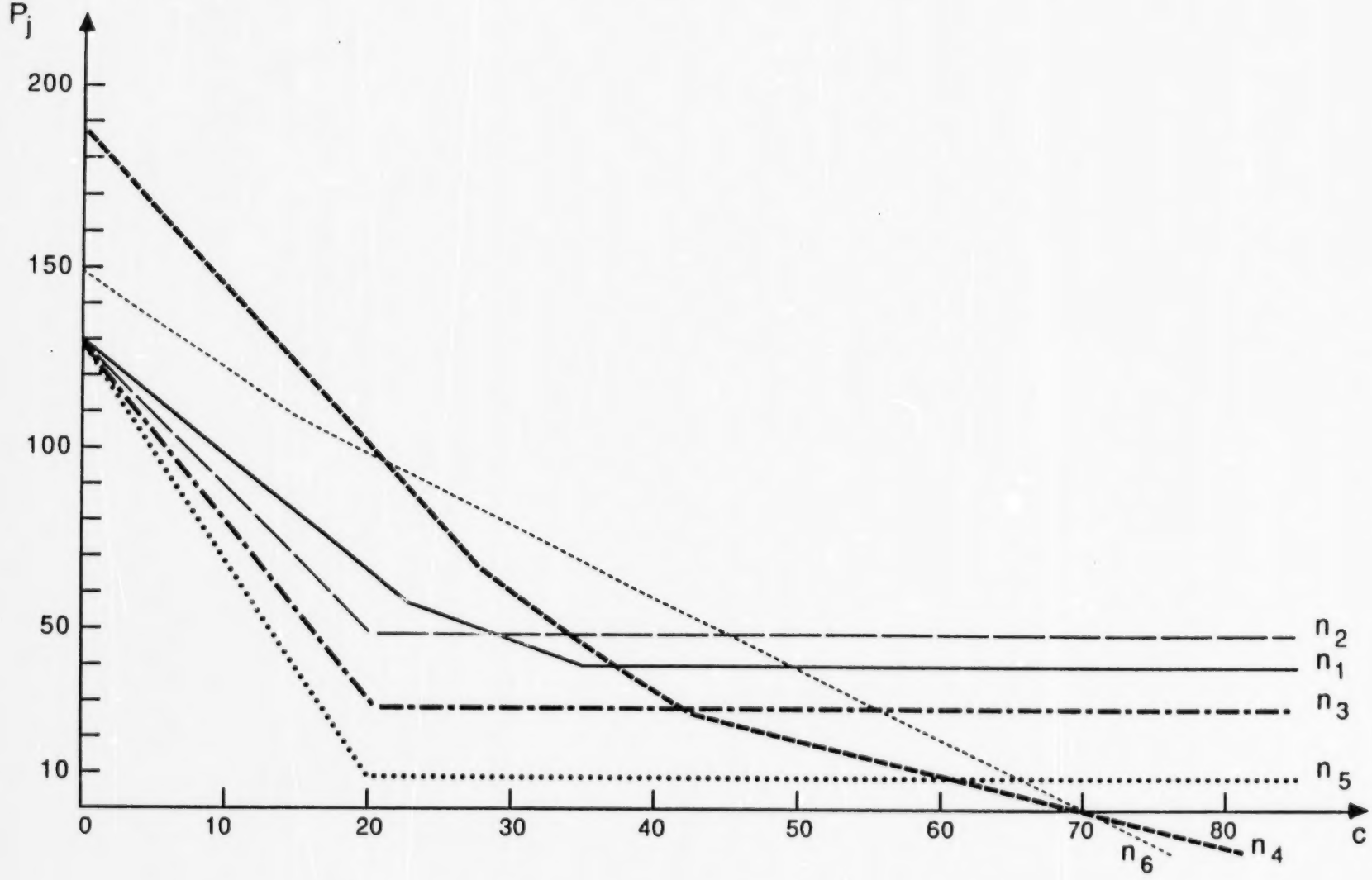

Figure 3 : Profit as a function of facility cost, using the first model

4. An Alternate Model and its Solution

The model described in the previous section may adequately describe the consumer behavior between the breakpoints but it is clearly unsatisfactory in their close vicinity. Why should all consumers at one demand point totally shun a facility just because they are slightly more attracted to it than to a different facility? It appears more reasonable to assume that the ratios of attractions of two facilities at some demand point approximate the ratio of purchases made at the two respective facilities. Generalizing this behavioral assumption to all given facilities we can define $p_{ij} = \phi_{ij} / \sum_{k \in N} \phi_{ik}$ as the proportion of the demand at node n_i which is to be satisfied by the facility located at n_j. Note that this attraction function is a special case of that used by Huff (1973). If the new facility shares its location with an already existing one, denote the new facility by j' and the old one by j, then define $p_{jj'} = w_j'/(w_j + w_j')$. For notational convenience this case is not considered below in the derivation of optimality conditions. The demand at n_i satisfied by the facility at n_j is $b_i\ p_{ij}$ and the total demand satisfied by a facility at n_j is $\sum_{i \in N} b_i\ p_{ij}$. Using again the assumption that the demand is expressed in dollars and assuming a cost function linear in the weight of the new facility, we obtain the profit function of the new facility located at n_j as

$$P_j = \sum_{i \in N} b_i \frac{w_j}{w_j + d_{ij} \sum_{k \neq j} w_k/d_{ik}} - cw_j \qquad (10)$$

Setting the derivative $\frac{dP_j}{dw_j} = 0$ yields the optimality condition

$$\sum_{i \in N} b_i \frac{d_{ij} \sum_{k \neq j} w_k/d_{ik}}{(w_j + d_{ij} \sum_{k \neq j} w_k/d_{ik})^2} = c \qquad (11)$$

Relation (11) is a polynomial equation in w_j of order up to n^2. It can easily be proved that for given values of

$$c < \sum_{i \in N} \frac{b_i}{d_{ij} \sum_{k \neq j} w_k / d_{ik}} \tag{12}$$

a solution of (11) exists and is unique. If c violates condition (12), then the optimal weight is $\bar{w}_j = 0$, resulting in a zero profit. Let again (12) be satisfied. Then inserting (11) in (10) yields the profit corresponding to the optimal weight $\bar{w}_j$:

$$\bar{P}_j = \sum_{i \in N} b_i \frac{\bar{w}_j^2}{(\bar{w}_j + d_{ij} \sum_{k \neq j} \bar{w}_k / d_{ik})^2} \tag{13}$$

In order to obtain a closed form expression for the profit of optimum, $\bar{P}_j$ has been written as a function of $\bar{w}_j$ even though both $\bar{P}_j$ and $\bar{w}_j$ are functions of c, as a given value of c implies an optimal weight $\bar{w}_j$ as well as a profit at optimum $\bar{P}_j$. To illustrate, consider again the numerical example in Figure 1. The profit function for a tentative location at n_2 is

$$P_2 = 40 \frac{7w_2}{7w_2 + 24} + 50 + 30 \frac{2w_2}{2w_2 + 11} + 0 + 10 \frac{12w_2}{12w_2 + 49} + 0 - cw_2$$

which yields the optimality condition

$$\frac{6\ 720}{(7w_2 + 24)^2} + \frac{660}{(2w_2 + 11)^2} + \frac{5\ 880}{(12w_2 + 49)^2} = c$$

and the optimal profit

$$\bar{P}_2 = \frac{1\ 960\ \bar{w}_2^2}{(7\bar{w}_2 + 24)^2} + 50 + \frac{120\ \bar{w}_2^2}{(2\bar{w}_2 + 11)^2} + \frac{1\ 440\ \bar{w}_2^2}{(12\bar{w}_2 + 49)^2}$$

We are now able to apply the same procedure as in the previous section, i.e. repeat the above determination of the functions $P_j(c)$ for all admissible j and then find their upper envelope. This will again result in optimal decisions with respect to location and weight as well as the corresponding profit for all nonnegative cost labels. The profit curves and their upper envelopes are displayed in Figure 4.

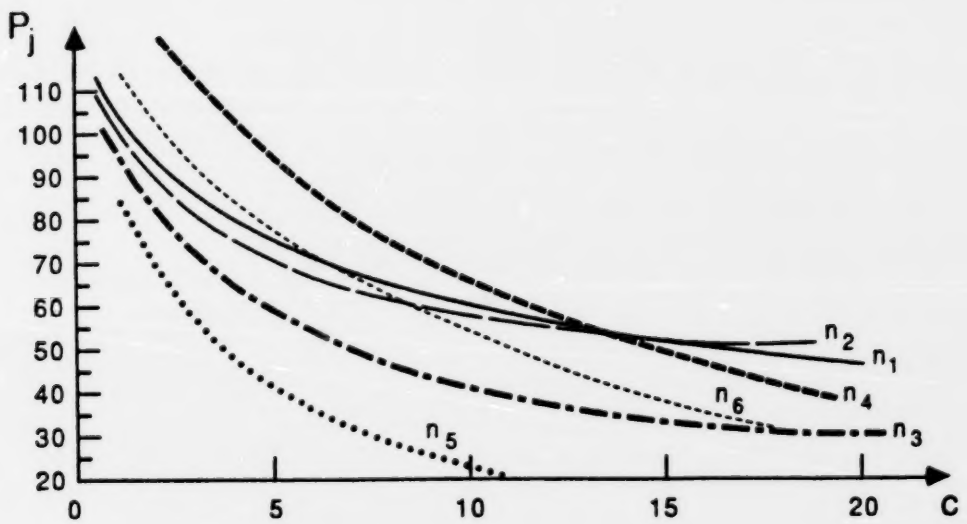

Figure 4 : Profit as a function of facility cost, using the second model

The corresponding decision rule is then:

if $c \in [0;13.64]$: locate at n_4,

if $c \in [13.64;14.14]$: locate at n_1, and

if $c \in [14.14; \infty[$: locate at n_2.

5. Conclusion

In this paper, a conditional location model on a network is discussed. Attraction and service functions are defined in terms of consumer-facility distance and facility weight; from these, influence spheres for the facilities can be defined. The optimal location and weight of a new facility are then determined, using two different models.

The model introduced in this paper can be extended in various ways. One possibility is a locational and weight game not unlike that described by Suzuki and Okabe (1987) for equally weighted facilities in the plane. Each facility optimizes its weight and location based on the current situation as shown previously. This procedure is then repeated until either an equilibrium is reached or it becomes clear that no equilibrium exists.

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